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A way to define the phase distribution for a single-mode quantum field

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Abstract. The idea that the phase of single-mode field may be correctly defined as a phase difference between the state considered and a highly excited coherent state treated as the reference phase state, although present in discussions about the quantum phase problem, have not been directly concretized and its consequences fully understood. In the present work we succeed in finding an effective mathematical procedure which corresponds exactly to this idea and so derive, using the results related to phase difference available in the literature, the phase distribution for a single-mode field. We discuss the obtained results.

The quest for a correct definition of the quantum phase variable, which began in the earliest days of quantum mechanics [1], still remains without definite results, although important progress has been made in recent years [2]. Most of the work in the field has been devoted to the properties of putative phase operators for single-mode quantum fields, or, equivalently, for a single harmonic oscillator. However, there are rigorous proofs that a bounded and self-adjoint phase operator satisfying canonical commutation relations with a number operator, on a dense domain which includes Hermite functions, does not exist [3, 4]. The absence of such a phase operator is usually ascribed to the semiboundedness of the eigenvalue spectrum of the number operator.

There are practical and physical reasons why one should treat the phase as a phase difference between the state considered and some reference phase state. As the variable canonically conjugated to the phase-difference operator is the number difference that is not bounded from below, the above reason for the non-existence of a phase operator is avoided in such an approach.

On this basis, in the paper of Luis and Sanchez-Soto [5] a new Hermitian operator which represents the phase difference between two fields of the same frequency ω was introduced for the first time and its essential properties discussed. Some further properties of this operator have been cleared up in the discussion between Pegg and Vaccaro [6] and Luis and Sanchez-Soto [7]. In the work of Yu [8] the peculiar properties of the spectrum of this operator were noticed and analysed and the phase distribution corresponding to the phase difference of two quantum states derived.

In this paper, using this distribution and fixing in it one state to be a coherent state and allowing its intensity to tend to infinity, we derive the phase distribution of a single quantum state. The main difficulty in this derivation was caused by the peculiar character of the spectrum of the phase-difference operator, which is everywhere dense but not continuous. This is

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probably the reason why the phase distribution for a single-mode field, based on the phase-difference operator, was not obtained earlier, although this possibility as an idea was present ever since the introduction of the phase-difference operator.

In order to present our results we shall first recapitulate the results we need from [5, 8].

In [5] a polar decomposition of the complex amplitudes of the two oscillators is represented as

$$a_1 a_2^\dagger = E_{12}\sqrt{N_2(N_1+1)} \tag{1}$$

where a_k, $a_k^\dagger$ and N_k ($k = 1, 2$), are annihilation, creation and number operators of the two oscillators, respectively. The operator E_{12} commutes with the total number operator $N = N_1 + N_2$, and may be written in the following form [5, 8]:

$$E_{12} = E_1^\dagger E_2 + \sum_{n=0}^{\infty} |0, n\rangle\langle n, 0| \mathrm{e}^{\mathrm{i}\phi(n)} \tag{2}$$

where the operator $E_k = (\sqrt{N_k+1})^{-1} a_k$ is the Susskind–Glogover exponential phase operator of the kth oscillator and where $|m, n\rangle = |m\rangle_1 \bigotimes |n\rangle_2$, $\langle m, n| = \langle m|_1 \bigotimes \langle n|_2$. The function $\phi(n)$ is an arbitrary real function defined on non-negative integers. Since E_{12} is unitary, a Hermitian phase-difference operator P_{12} can be defined by the relation $E_{12} = \exp(\mathrm{i}P_{12})$. The eigenvalues $\theta_{m,n}$ of the operator P_{12} are given by the expression [5]

$$\theta_{m,n} = \phi_0 + \frac{2m\pi}{n+1} \tag{3}$$

where $m = 0, 1, \ldots, n$; $n = 0, 1, 2, \ldots$ and ϕ_0 is a constant. The corresponding eigenstates

$$|\theta_{m,n}\rangle = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} \mathrm{e}^{\mathrm{i}k\theta_{m,n}} |n-k, k\rangle \tag{4}$$

form a complete orthonormal basis.

In the phase-space state basis (4), the phase-difference Hermitian operator P_{12} is simply given by

$$P_{12} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \theta_{m,n} |\theta_{m,n}\rangle\langle\theta_{m,n}|. \tag{5}$$

If, for simplicity, we choose the constant ϕ_0 to be zero, then the eigenvalues of the phase-difference operator are all rational numbers between 0 and 1 times 2π, and only these numbers, and each value is infinitely degenerated. For a rational number $q = m_0/n_0$, where integers m_0 and n_0 are prime to each other and $0 \leqslant m_0 \leqslant n_0$, the eigenstates corresponding to phase difference $2\pi q$ are

$$|q; s\rangle = \frac{1}{\sqrt{sn_0}} \sum_{k=0}^{sn_0-1} \mathrm{e}^{\mathrm{i}q2\pi k} |sn_0 - k - 1, k\rangle \tag{6}$$

where $s = 1, 2 \ldots$. When two harmonic oscillators are in a state $|\psi\rangle = \sum_{n,m=0}^{\infty} C_{m,n}|m, n\rangle$, with $\sum_{n,m=0}^{\infty} |C_{m,n}|^2 = 1$, it follows from (6) that the phase-difference distribution is given by [8]

$$P_\psi(2\pi q) = \sum_{s=1}^{\infty} \frac{1}{sn_0} \left| \sum_{k=0}^{sn_0-1} C^\star_{sn_0-1-k,k} \mathrm{e}^{\mathrm{i}2\pi qk} \right|^2. \tag{7}$$

Now, to define the phase distribution of a single oscillator we shall proceed as follows. We shall consider a state of two oscillators in which one oscillator is in the coherent state while the other one is in an arbitrary state $|\psi\rangle$:

$$|\psi\rangle = \sum_n A_n|n\rangle; \sum |A_n|^2 = 1. \tag{8}$$

In this case the phase distribution (7) takes the following form:

$$P(2\pi q) = \sum_{s=1}^{\infty} \frac{1}{sn_0} \left\{ \sum_{n,m=0}^{sn_0-1} A_n A_m^\star \mathrm{e}^{-\mathrm{i}2\pi q(n-m)} \langle sn_0 - n - 1|\alpha\rangle\langle\alpha|sn_0 - m - 1\rangle \right\} \tag{9}$$

where $\alpha = r\mathrm{e}^{\mathrm{i}\phi}$. For simplicity, hereafter we shall take $\phi = 0$.

The possibility of obtaining a phase distribution for a single oscillator from the phase-difference distribution (9) is based on the fact that states of a single harmonic oscillator with a well-defined phase exist. Such states are high-energy coherent states $|\alpha\rangle$, $|\alpha| \to \infty$. Now if we consider a state of two oscillators such that one oscillator is in the high-energy coherent state and the other one in an arbitrary state $|\psi\rangle$, then the phase-difference distribution (9) can be nothing else but the phase distribution of the state $|\psi\rangle$, since the high-energy coherent state has a well-defined phase.

For sufficiently high r we can approximate $\langle sn_0 - n - 1|\alpha\rangle$ in equation (9) by [9, 10]

$$\langle sn_0 - k - 1|\alpha\rangle \approx (2\pi r^2)^{-\frac{1}{4}} \exp\left[-\frac{(r^2 - sn_0 + k + 1)^2}{4r^2}\right]. \tag{10}$$

From this approximation we can conclude that a significant contribution to the summation over s gives only those values of s for which sn_0 is concentrated around r^2 within an interval of the order r. Due to this, as $1/sn_0$ varies slowly compared with the corresponding exponential factor we can fix it to $1/r^2$, and then approximate the sum by the corresponding integral which can be calculated easily so that we obtain after integration:

$$P(2\pi q) = \frac{1}{n_0 r^2}\left[\sum_{k,l} A_k A_l^\star \mathrm{e}^{-\mathrm{i}2\pi q(k-l)}\right]. \tag{11}$$

This is the probability for a fixed rational number q multiplied by 2π. For numbers which do not belong to this class the probability is zero because they do not belong to the spectrum of the phase-difference operator. As the spectrum of the phase-difference operator is not continuous, in order to find the probability in a small interval Δq around q we are not allowed to integrate this expression. Instead of integration we have to add all probabilities for all rational numbers which belong to this interval and contribute to this probability. In our case we can take only those rational numbers from the considered interval for which the denominator n_0 is no greater than r^2 because for greater n_0 such would be also sn_0 so that due to this, as we saw above, the probability would be negligibly small.

Now we can proceed as follows. When n_0 is such that $1/n_0 < \Delta q$ some rational numbers of the form m/n_0 will certainly belong to the considered interval Δq for some m. Obviously, there will be altogether $\Delta q/(1/n_0)$ such numbers. The denominator n_0 for various rational numbers which gives a relevant contribution to the probability distribution can approximately take r^2 different values. The interval Δq may always be chosen so small that the function in brackets may be treated as constant. Bearing all this in mind and taking the limit $r \to \infty$ we obtain the expression for the probability we were looking for:

$$\Delta q\left[\sum_{k,l} A_k A_l^\star \mathrm{e}^{\mathrm{i}2\pi q(k-l)}\right].$$

Evidently, from this formula, we can write for finite interval $[a, b]$

$$\sum_{q\epsilon[a,b];q\epsilon Q} P(2\pi q) = \int_a^b \left[\sum A_k A_l^\star \mathrm{e}^{\mathrm{i}2\pi x(k-l)}\right] \mathrm{d}x. \tag{12}$$

It is obvious from the last equation that the expression $[\sum A_k A_l^\star \mathrm{e}^{\mathrm{i}2\pi x(k-l)}]$ may be interpreted as the phase probability distribution.

We have thus shown that the phase distribution of a single-oscillator state can be derived from the phase-difference distribution. It is easily seen that this phase distribution is almost identical with the phase distribution which is usually referred to as the canonical phase distribution [11]. More precisely, for the probability in any fixed finite interval they both give the same numerical result. But there is one subtle difference between the two phase distributions. Namely, the canonical phase distribution is a continuous function of its argument in a 2π interval, while the present distribution assumes non-zero values only for those values of q which are rational numbers. This subtle difference may be of theoretical interest but, at present, it does not seem to have practical importance.

Although the expression (12) for the single-oscillator phase distribution, appeared before, e.g. in [11], the derivation presented there was based on general postulates and the connection with specific experiments and models was not so clear. In the present work this distribution is derived in a direct way which emphasizes the essential role of the infinite-energy (hence classical) limit of a coherent state as a reference state of a well-defined phase, necessary for promoting a phase-difference distribution of two oscillators into a phase distribution of a single oscillator. It should be also noted that the phase-difference operator of Luis and Sanchez-Soto [5], upon which our results essentially rely, is a realization in a modified form of Dirac's original idea to find the phase operator using a polar decomposition of creation and annihilation operators. These facts should be considered as merits of the considered distribution.

In the absence of a unique pre-eminent phase operator and phase distribution it is, in our opinion, important to elucidate various relations between those in use and to supply physical and methodological grounds for them, whenever possible. Hopefully, this will enable us, if not to find the unique correct one, then at least to establish some reasonable hierarchy between them.

We believe that our recent papers [12, 13], and the present one contribute in a way to this goal.

Recently, Luis and Sanchez-Soto [14, 15] have developed and discussed at length a procedure giving the phase-difference probability-distribution function for a two-mode field in terms of the single-mode field phase distribution. For the latter they used and discussed distribution functions from various approaches.

When the single-mode phase distribution is taken to be the one obtained in Pegg–Barnet formalism, their results may be reproduced almost trivially in the present approach, if one takes into account the character of the spectrum of the phase-difference operator and applies the above procedure for summation of probabilities over the spectrum.

Namely, the probability that one field has phase θ and, simultaneously, that the other has phase $\theta + \phi$, is given by

$$P_1(\theta) P_2(\theta + \phi) \tag{13}$$

where P_1 and P_2 are corresponding single-mode probabilities relative to the same reference state, and θ and $(\theta + \phi)$ belong to the spectra of corresponding phase-difference operators. Obviously, this expression may be interpreted as the probability that the phase difference is ϕ when θ is fixed.

Now, taking one and the same coherent state as a reference state for both single-mode fields, using our equation (11) we can write

$$P_1(\theta) P_2(\theta + \phi) = \frac{1}{n_0 r^2} \left[\sum_{k,l} A_k A_l^\star \mathrm{e}^{-2\pi q_1 (k-l)} \right] \frac{1}{n_0' r^2} \left[\sum_{m,n} A_m' A_n'^\star \mathrm{e}^{-2\pi q_2 (m-n)} \right] \tag{14}$$

where θ and $(\theta + \phi)$ are represented as $2\pi q_1$ and $2\pi q_2$ respectively, and where the other quantities corresponding to the second mode are denoted by primes. Applying to both factors in (14) the same procedure which led from equation (11) to (12), i.e. adding probabilities for all rational numbers which belong to the small intervals $\Delta\theta$ and $\Delta\phi$ around θ and ϕ respectively, we obtain for the probability distribution of phase difference $f(\phi)$

$$f(\phi) = \int \mathrm{d}\theta \, \langle \theta + \phi, \theta | \rho | \theta + \phi, \theta \rangle. \tag{15}$$

Here $|\theta_1, \theta_2\rangle = |\theta_1\rangle \bigotimes |\theta_2\rangle$ and $|\theta\rangle$ is the standard Susskind–Glogover phase state. The last expression is identical with expression (3.11) from [14], but here it is derived in a transparent and straightforward way with an evident probabilistic interpretation and some difficulties avoided.

It should be noted that the phase-difference distribution of the two states, obtained from the phase-difference distribution of these states relative to the same coherent state, is not the same as the phase difference distribution of these states calculated relative to each other. The reason for this, as explained in [8], is the fact that the phase-difference operator cannot be represented as the difference of corresponding phase operators because such Hermitian phase operators do not exist. However, 'the close resemblance of these expressions' was found in [14], where the expressions for these two distributions were discussed.

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